# Technical Comments

## Comment on "A Paradoxical Case in a Stability Analysis"

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PARNES[1] has analyzed a specific beam buckling problem and discovered a paradox in that, for given conditions, the buckling load may increase with an increase in the overall length of the structural member. The author has not checked these analyses but, by considering a similar but simpler model, is able to show that the "paradox" is not unique and may be imaginary.

Two vertical bars $AB$ and $BC$ are initially in line and pinned at the points $A$, $B$, and $C$. A compressive vertical load $P$ acts at the point $C$, which may move only vertically, and a lateral spring of stiffness $\beta$ acts at the common pinned joint $B$ (Fig. 1).

This model is a simple modification of those used by Timoshenko and Gere[2] to demonstrate the energy method in buckling analyses and is a simpler form of that used by Parnes,[1] with the rigid member $AB$ and the lateral spring replacing his elastic member $AB$. For a small angular displacement $\alpha$ of $AB$, the lateral displacement of the spring is $\alpha L$, and the downward vertical movement of point $B$ is $\alpha^2 L/2$.

The overall downward vertical movement of point $C$ is therefore

$$\delta = \tfrac{1}{2}\alpha^2 L[1 + (1/c)] \tag{1}$$

and equating the strain energy in the spring to the work done by $P$ gives

$$P(\alpha^2 L/2)[1 + (1/c)] = \tfrac{1}{2}\beta(\alpha L)^2 \tag{2}$$

Therefore, the critical value of $P$ is given by

$$P = \beta L/[1 + (1/c)] \tag{3}$$

and, as $c$ increases from zero through unity to infinity, one obtains the results shown in Table 1. Thus as $c$ increases, i.e., overall length of the structure increases, the critical load also increases.

This is not a paradox, however, as can be shown easily by reconsidering the basic physical model. When $c$ is infinite, the applied load $P$ is transmitted to member $AB$ as a vertical load $P$ at $B$, and the critical load is $P/\beta L = 1$, as in Ref. 2. As $c$ decreases to, say, 1, the equivalent force in member $CB$, acting at $B$, for a small lateral displacement of $B$, must be greater than $P$ even though the vertical component is still $P$, and this results in a decrease in critical load $P$. Examination of Eq. (1) also shows that the downward movement of point $C$ increases significantly as $c$ decreases, resulting in a corresponding decrease in the critical load.

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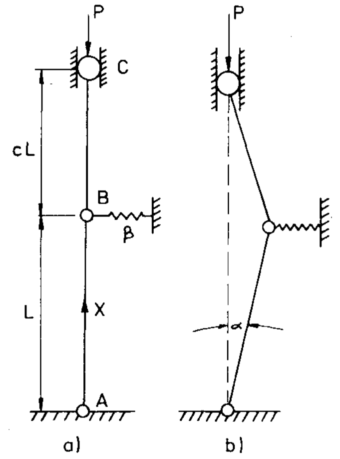


**Fig. 1 Geometry of problem.**

**Table 1 Relationship of $c$ to critical load**

| $c$ | 0 | 0.5 | 1 | 2 | 3 | 4 | $\alpha$ |
|---|---|---|---|---|---|---|---|
| $P/\beta L$ | 0 | 0.3 | 0.5 | 0.6 | 0.75 | 0.8 | 1 |

The limiting condition at $c \to 0$, giving $P/\beta L \to 0$, is subject to the proviso that, if member $BC$ exists and has zero length ($c = 0$), then there can be no instability, as point $B$ of member $AB$ will be constrained from moving laterally. Thus $P_{crit} \to \alpha$ and is not zero. This apparent paradox is thus explained.

### References

[1] Parnes, R., "A Paradoxical Case in a Stability Analysis," *AIAA Journal*, Vol. 15, Oct. 1977, pp. 1533-1534.
[2] Timoshenko, S. and Gere, J., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961, pp. 83-86.

## Reply by Author to D.J. Johns

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THE author notes that Dr. Johns, in considering a simple system consisting of rigid bars, has presented another model for which increased stability is obtained with increased length of a member. Certainly, no claim was made by the author that the model considered in the original paper represented a unique case.

The physical reason for which this phenomenon occurs in the two cases cited is explained by Dr. Johns in terms of an equivalent force. A more precise physical explanation may, perhaps, be given as follows. In both models, member $BC$ can transmit only a force concurrent with the chord $BC$ of the deflected position. Consequently, for any given small

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deflection $\Delta$ of point $B$, the total force transmitted to member $AB$ by member $BC$ consists of a vertical component $P$ and a transverse component $R = P\Delta/cL$. Thus, for decreasing values of $c$, member $AB$ is subjected to a vertical force $P$ in addition to an increasing transversal force varying inversely with $c$. This therefore results in a decreasing buckling load with decreasing length $cL$.

As for the behavior when $c \to 0$, the solution curves given in Fig. 2 of the original paper are valid for all $0 < c$. It was pointed out originally that when $c = 0$ identically, the buckling load is in fact that of a fixed-hinged member [see Eq. (6)] and not zero. This was indicated most clearly by demonstrating that the relevant root of the transcendental equation is given by point $A$ of Fig. 3, when $c = 0$. Dr. Johns has failed to notice this point, since his dicussion is based on the implication that the solution curves of Fig. 2 are meant to include the value of $c = 0$.

It is a well-known phenomenon in structural mechanics that, in a general solution, when two hinges are made to coincide (here $c = 0$), a singular behavior usually will occur, and in the paper a mathematical explanation of this isolated case was given. In no sense, then, did the meaning of the paradox apply to this singular case.

Be means of his example, Dr. Johns confirms that it is possible, and clearly not in an imaginary sense, to obtain increasing instability loads for members of increasing length with values $c > 0$. Although for two-dimensional plate problems it is known that increasing a dimension can result in increased stability,[1] it is rather unusual, if not paradoxical, that for a one-dimensional case the same phenomenon occurs when a member is subjected to axial compressive loads.

### References

[1] Timoshenko, S. and Gere, T., *Theory of Elastic Stability*, McGraw-Hill, New York, 1961, pp. 351 ff.

# Comment on "A Paradoxical Case in a Stability Analysis"

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PARNES[1] states with respect to column members that "it is generally accepted that the load at which instability occurs decreases as the member increases in length." He terms "paradoxical" a specific model that he presents in which the opposite is true. However, even in the standard text on elastic stability by Timoshenko and Gere,[2] models of such "paradoxical" elastic systems are presented. In addition, structural models having such properties have been used to study nonlinear buckling phenomena by Panovko and Gubanova.[3]

The explanation of the supposed paradox is that, if concentrated elastic restraint against displacement occurs at the ends of a rigid or relatively rigid member that is axially loaded, the rotational stiffness of the member is the one pertinent to elastic stability, and this increases with length of the member. One system considered by Timoshenko and Gere, shown in Fig. 1, consists of an infinitely rigid bar of length $\ell$ hinged at one end and laterally restrained by a spring of spring constant $K$ at the other end. The axial load at which buckling occurs is given as

$$P = K\ell \tag{1}$$

which is the condition for neutral equilibrium of horizontal forces when the bar undergoes a small angular rotation $\alpha$. Other more complex cases of pin-connected rigid bars under lateral restraints are also given by Timoshenko and Gere and also lead to buckling loads that increase with length.

In Parnes' first model, shown in Fig. 2, the linear spring of Fig. 1 is effectively replaced by a cantilever beam-column member $AB$ of stiffness $EI$ and length $L$, restraining the lateral motion of the rigid link $BC$, of length $CL$ and also carrying though the same compressive load $P$ as imposed on the rigid link. The formula for lateral tip deflection of a cantilever beam-column, $y_t$, under lateral force $F$ and axial load $P$, is[4]

$$y_t = \frac{FL}{P}\left(\frac{\tan\beta L}{\beta L} - 1\right) \tag{2}$$

where $\beta^2 = P/EI$, so that the stiffness $K(P)$ is

$$K(P) = F/y_t = P/L\left(\frac{1}{\tan\beta L/\beta L - 1}\right) \tag{3}$$

For small values of $P/EI$ (i.e., well below the buckling value of $\pi^2/4L^2$),

$$\frac{\tan\beta L}{\beta L} - 1 \approx 1 + \frac{(\beta L)^2}{3} - 1 \approx \frac{(\beta L)^2}{3} \tag{4}$$

or

$$K(P) \approx \frac{P}{L}\,\frac{1}{(PL^4/3EI)} = \frac{3EI}{L^3} \tag{5}$$

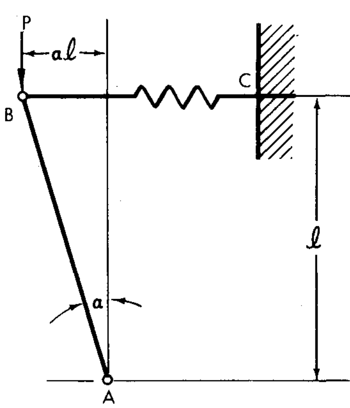


**Fig. 1 Stability problem with rigid links.**

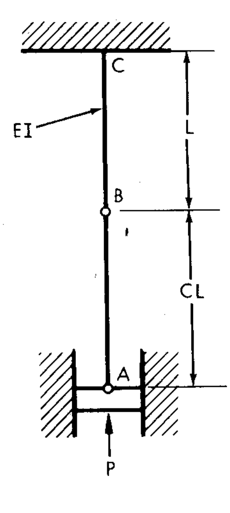


**Fig. 2 Stability problem with flexible links.**

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